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On non-symmetric vibration of deep spherical sandwich shells

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SUMMARY

This paper deals with the free non-symmetric vibration of deep spherical sandwich shells. The sandwich shell considered herein consists of three layers. A variational technique is utilized to obtain the equations of motion as well as the appropriate boundary conditions. The effects of transverse shear deformation and rotary inertia have been included in this analysis.

New deformation functions have been introduced which considerably simplify the system of differential equations. The final solution is obtained in terms of Legendre functions.

Numerical computations have been performed for the symmetric case and graphs are included to show the frequency variation with ϕ and h/R for various modes.

1. Introduction

In the past, free vibration of shallow and deep shells has been studied quite extensively by many authors [1, 2, 3]. On the contrary, little work has been published on the vibration of spherical sandwich shells [4, 5, 6]. The work by Koplik and Yu [4] deals with the axisymmetric vibrations of spherical sandwich caps using associated variational equations of motion. Reference [5] deals with the transverse vibrations of a three layered shallow spherical sandwich shell. The problem of the axisymmetric vibration of deep spherical sandwich shells was considered by the authors, and has been reported earlier [6]. In reference [6] the face sheets have been taken as membranes.

In this paper, the normal mode of the non-symmetric vibration of a deep sandwich spherical shell is studied. The sandwich shell consists of three layers. The facings are assumed to be made of isotropic elastic material and the core is assumed to be a low density and low strength material. The formulation of the problem is based on the linear strain–displacement relationship and a variational principle [7] is utilized to obtain the stress–displacement relationships, the equations of motion as well as the appropriate boundary conditions. The effects of transverse shear deformation and rotary inertia have been included in this analysis. The core is assumed to be incompressible in the radial direction and its face parallel stresses are neglected. The flexural rigidity of the facings are also taken into account.

The system of differential equations is reduced to a considerably simplified form by introducing new variables which are functions of the displacement components. These arbitrary functions are expressed in terms of associated Legendre functions as described elsewhere in this paper. The elastic and geometric parameters have been expressed in non-dimensional form. The numerical values of the nondimensionalized frequency parameter Ω for the clamped edge sandwich shells are presented. The graphs show the effects of flexural rigidity of the face sheets, opening angle ϕ_o and the geometric and elastic properties of the core on the frequency.

2. Notations and fundamental relationships

The displacement components at any arbitrary point of the i-th layer, u_i', v_i' and w_i' in θ, ϕ, and r directions respectively are assumed to be

$$u_i' = u_i + Z_i \beta_{\theta i}, \quad v_i' = v_i + Z_i \beta_{\phi i}, \quad w_i' = w_i . \tag{2.1}$$

Here, u_i, v_i and w_i are the mid-surface displacement components in the i-th layer; β_θ and β_ϕ are changes of slope of the normal to the middle surface from where Z_i is measured in the outward radial direction. The strain-components in the spherical coordinate system are written as

$$(1+Z_i/R_i)\varepsilon'_{\phi i}=\varepsilon_{\phi i}+Z_i k_{\phi i}\,,\quad (1+Z_i/R_i)\varepsilon'_{\theta i}=\varepsilon_{\theta i}+Z_i k_{\theta i}\,,$$
$$(1+Z_i/R_i)\gamma'_{\theta\phi i}=\gamma_{\theta\phi i}+Z_i k_{\theta\phi i}\,,\quad (1+Z_i/R_i)\gamma'_{r\phi i}=\gamma_{r\phi i}\,,\quad (1+Z_i/R_i)\gamma'_{r\theta i}=\gamma_{r\theta i}\,. \tag{2.2}$$

In eqs. (2.2), R_i is the radius of the middle surface of the layer and $\varepsilon'_{\phi i}$, $\varepsilon'_{\theta i}$, $\gamma'_{\theta\phi i}$, $\gamma'_{r\phi i}$, $\gamma'_{r\phi i}$ are strains at any arbitrary point. The mid-surface strains $\varepsilon_{\phi i}$, $\varepsilon_{\theta i}$, $\gamma_{\theta\phi i}$, $\gamma_{r\phi i}$, $\gamma_{r\theta i}$ are expressed in terms of mid-surface displacements as follows

$$R_i\varepsilon_{\phi i}=v_{i,\phi}+w_i\,,\quad R_i\varepsilon_{\theta i}=\operatorname{cosec}\phi\, u_{i,\theta}+\cot\phi\, v_i+w_i\,,$$
$$R_i\gamma_{\theta\phi i}=u_{i,\phi}-\cot\phi\, u_i+\operatorname{cosec}\phi\, v_{i,\theta}\,,\quad R_i\gamma_{r\phi i}=w_{i,\phi}-v_i+R_i\beta_{\phi i}\,,$$
$$R_i\gamma_{r\theta i}=\operatorname{cosec}\phi\, w_{,\theta}-u_i+R_i\beta_{\theta i}\,,\quad R_i k_{\phi i}=\beta_{\phi i,\phi}\,,$$
$$R_i k_{\theta i}=\operatorname{cosec}\phi\,\beta_{\theta i,\theta}+\cot\phi\,\beta_{\phi i}\,,\quad R_i k_{\theta\phi i}=\beta_{\theta i,\phi}-\cot\phi\,\beta_{\theta i}+\operatorname{cosec}\phi\,\beta_{\phi i,\theta}\,. \tag{2.3}$$

In the above equations, the notation $w_{i,\theta}$ stands for the partial derivative of w_i with respect to θ.

The normal stresses $\sigma'_{\phi i}$, $\sigma'_{\theta i}$ and shear stresses $\tau'_{r\theta i}$, $\tau'_{r\phi i}$, $\tau'_{\theta\phi i}$ are expressed in terms of the stress resultants $N_{\phi i}$, $N_{\theta i}$, $N_{\theta\phi i}$, moment resultants $M_{\phi i}$, $M_{\theta i}$, $M_{\theta\phi i}$, and shear stress resultants $Q_{\phi i}$ and $Q_{\theta i}$ in the following manner [7]

$$(1+Z_i/R_i)\sigma'_{\phi i}=N_{\phi i}/h_i+Z_i M_{\phi i}/I_i\,,$$
$$(1+Z_i/R_i)\tau'_{\theta\phi i}=N_{\theta\phi i}/h_i+Z_i M_{\theta\phi i}/I_i\,,$$
$$\begin{aligned}(1+Z_i/R_i)\tau'_{r\phi i}=&\tfrac{3}{2}\{1-(2Z_i/h_i)^2\}Q_{\phi i}/h_i-\\&-\{p^+_{\phi i}(1+h_i/2R_i)[1-2(2Z_i/h_i)-3(2Z_i/h_i)^2]+\\&+p^-_{\phi i}(1-h_i/2R_i)[1+2(2Z_i/h_i)-3(2Z_i/h_i)^2]\}/4\,.\end{aligned} \tag{2.4}$$

where h_i is the thickness of the i-th layer and the symbol $I_i=h_i^3/12$. The quantities $p^+_{\phi i}$, $p^-_{\phi i}$, $p^+_{\theta i}$, $p^-_{\theta i}$ are given as

$$\tau'_{r\phi i}|_{Z_i=h_i/2}=p^+_{\phi i},\ \tau'_{r\phi i}|_{Z_i=-h_i/2}=p^-_{\phi i},\ \text{etc.} \tag{2.5}$$

The equations for $\sigma'_{\theta i}$ and $\tau'_{r\theta i}$ can be obtained by replacing θ for ϕ in eqs. (2.4)a and (2.4)c.

3. Equations for sandwich spherical shells

The spherical sandwich shell considered in this investigation consists of three layers. The two face layers are made of isotropic elastic material. The thickness of each face sheet is h and its modulus of elasticity and Poisson's ratio are E and ν respectively. The change in slopes of the

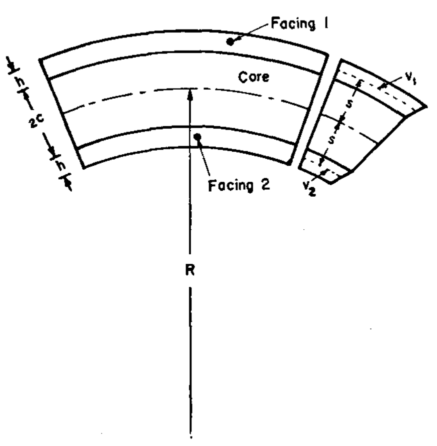


Figure 1. Spherical sandwich shell element with the distribution of deformation.

normals to the middle surfaces are assumed to be equal in both face sheets, and are given by β_ϕ and β_θ in the ϕ and θ directions respectively. The thick layer between the face sheets is a low density and low strength material. The compression or stretching of the core material in the radial direction is assumed to be negligible. The face parallel stresses in the core are negligible as compared to those in the face sheets. Thus,

$$\sigma_\phi = \sigma_\theta = \tau_{\phi\theta} = 0\,. \tag{3.1}$$

The displacement components are taken in the following form

$$u = \tilde{u} + (\bar{u} - h\beta_\theta/2)Z/c\,, \quad v = \tilde{v} + (\bar{v} - h\beta_\phi/2)Z/c\,, \tag{3.2}$$

where $2c$ is the thickness of the core, and the displacement components $\tilde{u}$, $\tilde{v}$, $\bar{u}$, $\bar{v}$ are given by

$$\tilde{u} = (u_1 + u_2)/2\,, \quad \tilde{v} = (v_1 + v_2)/2\,, \quad \bar{u} = (u_1 - u_2)/2\,, \quad \bar{v} = (v_1 - v_2)/2\,. \tag{3.3}$$

For convenience, the stress resultants and moment resultants are defined as below

$$\begin{aligned} &\tilde{N}_\phi = (N_{\phi 1} + N_{\phi 2})/2\,, \quad \bar{N}_\phi = (N_{\phi 1} - N_{\phi 2})/2\,, \quad \tilde{N}_\theta = (N_{\theta 1} + N_{\theta 2})/2\,, \\ &\bar{N}_\theta = (N_{\theta 1} - N_{\theta 2})/2\,, \quad \tilde{M}_\phi = (M_{\phi 1} + M_{\phi 2})/2\,, \quad \bar{M}_\phi = (M_{\phi 1} - M_{\phi 2})/2\,, \text{ etc.} \end{aligned} \tag{3.4}$$

With the introduction of the strains, stress-resultants and displacements as given in eqs. (2.2), (2.4), (3.2)–(3.4), the total energy in the shell element can be written as

$$\begin{aligned} &[2R(\tilde{N}_\phi\tilde{\varepsilon}_\phi + \bar{N}_\phi\bar{\varepsilon}_\phi + \tilde{N}_\theta\tilde{\varepsilon}_\theta + \bar{N}_\theta\bar{\varepsilon}_\theta + \tilde{N}_{\theta\phi}\tilde{\gamma}_{\theta\phi} + \bar{N}_{\theta\phi}\bar{\gamma}_{\theta\phi} + \tilde{Q}_\phi\tilde{\gamma}_{r\phi} + \bar{Q}_\phi\bar{\gamma}_{r\phi} \\ &\qquad + \tilde{Q}_\theta\tilde{\gamma}_{r\theta} + \bar{Q}_\theta\bar{\gamma}_{r\theta} + \tilde{M}_\phi k_\phi + \tilde{M}_\theta k_\theta + \tilde{M}_{\theta\phi}k_{\theta\phi}) + \\ &+ 2s(\tilde{N}_\phi\bar{\varepsilon}_\phi + \bar{N}_\phi\tilde{\varepsilon}_\phi + \tilde{N}_\theta\bar{\varepsilon}_\theta + \bar{N}_\theta\tilde{\varepsilon}_\theta + \tilde{N}_{\theta\phi}\bar{\gamma}_{\theta\phi} + \bar{N}_{\theta\phi}\tilde{\gamma}_{\theta\phi} + \tilde{Q}_\phi\bar{\gamma}_{r\phi} + \bar{Q}_\phi\tilde{\gamma}_{r\phi} + \\ &\qquad + \tilde{Q}_\theta\bar{\gamma}_{r\theta} + \bar{Q}_\theta\tilde{\gamma}_{r\theta} + \bar{M}_\phi k_\phi + \bar{M}_\theta k_\theta + M_{\theta\phi}k_{\theta\phi}) + \\ &+ \rho h R^2\{s_1(\tilde{u}_{,t}^2 + \tilde{v}_{,t}^2 + w_{,t}^2) + s_2(\bar{u}_{,t}^2 + \bar{v}_{,t}^2 + 2s_3(\tilde{u}_{,t}\bar{u}_{,t} + \tilde{v}_{,t}\bar{v}_{,t}) + \\ &\qquad + 2s_4R(\tilde{u}_{,t}\beta_{\theta,t} + v_{,t}\beta_{\phi,t}) + 2s_5R(\bar{u}_{,t}\beta_{\theta,t} + \bar{v}_{,t}\beta_{\phi,t}) + s_6R^2(\beta_{\theta,t}^2 + \beta_{\phi,t}^2)\} - \\ &\{\Lambda_1 + 12(1+v)(\tilde{Q}_\phi^2 + \tilde{Q}_\theta^2 + \overline{Q_\phi^2} + \overline{Q_\theta^2})/5\}R^2(1 + s^2/R^2)/Eh - \\ &\{\Lambda_2 + 12(1+v)(\tilde{Q}_\phi\bar{Q}_\phi + \tilde{Q}_\theta\bar{Q}_\theta)/5\}4sR/Eh - \Lambda_3R^2(1 + s^2/R^2)/EI - \\ &\Lambda_4 4Rs/EI + R(Q_{\phi c}\gamma_{r\phi c} + Q_{\theta c}\gamma_{r\theta c}) - (Q_{\phi c}^2 + Q_{\theta c}^2)3R^2/10G_c c - \\ &\{-r_G(1 + sc/R^2)(\tilde{Q}_\phi\tilde{p}_\phi + \bar{Q}_\phi\bar{p}_\phi + \tilde{Q}_\theta\tilde{p}_\theta + \bar{Q}_\theta\bar{p}_\theta) - \\ &\quad - r_G(s/R + c/R)(\bar{Q}_\phi\tilde{p}_\phi + \bar{Q}_\theta\tilde{p}_\theta + \tilde{Q}_\phi\bar{p}_\phi + \tilde{Q}_\theta\bar{p}_\theta) - (Q_{\phi c}\tilde{p}_\phi + Q_{\theta c}\tilde{p}_\theta) - \\ &\quad - (Q_{\phi c}\bar{p}_\phi + \bar{Q}_{\theta c}\bar{p}_\theta)c/R + [2r_Gh(1 + c^2/R^2)/3 + c(1 + \tfrac{5}{3}c^2/R^2)](\tilde{p}_\phi^2 + \tilde{p}_\theta^2) + \\ &+ [2r_Gh(1 + c^2/R^2)/3 + c(\tfrac{5}{3} + c^2/R^2)](\bar{p}_\phi^2 + \bar{p}_\theta^2) + (\tilde{p}_\phi\bar{p}_\phi + \tilde{p}_\theta\bar{p}_\theta)(r_G + 2r_h)8ch/3R\}]\sin\phi\, d\phi\, d\theta\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &s_1 = 1 + s^2/R^2 + h^2/12R^2 + r_\rho r_h(1 + c^2/3R^2)\,, \\ &s_2 = 1 + s^2/R^2 + h^2/12R^2 + r_\rho r_h(1 + 3c^2/5R^2)/3\,, \\ &s_3 = 2s/R + 2r_\rho r_h c/3R\,, \\ &s_4 = (1 - 2r_\rho r_h^2)h^2/6R^2\,, \\ &s_5 = \{sh^2/R^3 - r_\rho r_h(1 + 3c^2/5R^2)h/R\}/6\,, \\ &s_6 = \{1 + s^2/R^2 + 3h^2/20R^2 + r_\rho r_h(1 + 3c^2/5R^2)\}h^2/12R^2\,, \\ &r_\rho = \rho_c/\rho\,, \quad r_h = c/h\,, \quad r_G = G_c/G\,. \end{aligned} \tag{i}$$

G and v are the modulus of rigidity and Poisson's ratio of the face sheet; ρ is a mass density of face sheet; G_c and ρ_c are modulus of rigidity and mass density of the core material and t represents time.

Also,

$$\tilde{\varepsilon}_\phi = \tilde{v}_{,\phi} + w, \; \bar{\varepsilon}_\phi = \bar{v}_{,\phi}, \quad \tilde{\varepsilon}_\theta = \operatorname{cosec}\phi\, \tilde{u}_{,\theta} + \cot\phi\, \tilde{v} + w,$$
$$\bar{\varepsilon}_\theta = \operatorname{cosec}\phi\, \bar{u}_{,\theta} + \cot\phi\, \bar{v}, \quad \tilde{\gamma}_{\theta\phi} = \tilde{u}_{,\phi} + \cot\phi\, \tilde{u} + \operatorname{cosec}\phi\, \tilde{v}_{,\theta},$$
$$\bar{\gamma}_{\theta\phi} = \bar{u}_{,\phi} - \cot\phi\, \bar{u} + \operatorname{cosec}\phi\, \bar{v}_{,\theta}, \quad \tilde{\gamma}_{r\phi} = w_{,\phi} - \tilde{v} + R\beta_\phi, \quad \bar{\gamma}_{r\phi} = -\bar{v} + s\beta_\phi,$$
$$\tilde{\gamma}_{r\theta} = \operatorname{cosec}\phi\, w_{,\theta} - \bar{u}\, R\beta_\theta, \quad \bar{\gamma}_{r\theta} = -\bar{u} + s\beta_\theta, \quad \gamma_{r\phi c} = w_{,\phi} - \tilde{v} + \bar{v}R/c - \beta_\phi Rh/2c,$$
$$\gamma_{r\theta c} = \operatorname{cosec}\phi\, w_{,\theta} - \tilde{u} + \bar{u}R/c - \beta_\theta Rh/2c,$$
$$\Lambda_1 = \tilde{N}_\phi^2 + \bar{N}_\phi^2 + \tilde{N}_\theta^2 + \bar{N}_\theta^2 + 2(1+\nu)(\tilde{N}_{\theta\phi}^2 + \bar{N}_{\theta\phi}^2) - 2\nu(\tilde{N}_\phi \tilde{N}_\theta + \bar{N}_\phi \bar{N}_\theta),$$
$$\Lambda_2 = \tilde{N}_\phi \bar{N}_\phi + \tilde{N}_\theta \bar{N}_\theta + 2(1+\nu)\tilde{N}_{\theta\phi}\bar{N}_{\theta\phi} - \nu(\tilde{N}_\phi \bar{N}_\theta + \bar{N}_\phi \tilde{N}_\theta),$$
$$\Lambda_3 = \tilde{M}_\phi^2 + \bar{M}_\phi^2 + \tilde{M}_\theta^2 + \bar{M}_\theta^2 + 2(1+\nu)(\tilde{M}_{\theta\phi}^2 + \bar{M}_{\theta\phi}^2) - 2\nu(\tilde{M}_\phi \tilde{M}_\theta + \bar{M}_\phi \bar{M}_\theta),$$
$$\Lambda_4 = \tilde{M}_\phi \bar{M}_\phi + \tilde{M}_\theta \bar{M}_\theta + 2(1+\nu)\tilde{M}_{\theta\phi}\bar{M}_{\theta\phi} - \nu(\tilde{M}_\phi \bar{M}_\theta + \bar{M}_\phi \tilde{M}_\theta). \tag{3.6}$$

Considering the variation of displacement components and stress and moment resultants [7, 9], the constitutive equations for the sandwich shell and the proper boundary conditions are obtained. These differential equations of motion are given by

$$\operatorname{cosec}\phi\, \tilde{N}_{\theta,\theta} + \tilde{N}_{\theta\phi,\phi} + 2\cot\phi\, \tilde{N}_{\theta\phi} + \tilde{Q}_\theta + Q_{\theta c}/2 + (\operatorname{cosec}\phi\, \bar{N}_{\theta,\theta} + \bar{N}_{\theta\phi,\phi} + 2\cot\phi\, \bar{N}_{\theta\phi} + \bar{Q}_\theta)s/R = \rho hR(s_1\tilde{u} + s_3\bar{u} + s_4 R\beta_\theta)_{,tt},$$

$$\begin{aligned}&\tilde{N}_{\phi,\phi} + \operatorname{cosec}\phi\, \tilde{N}_{\theta\phi,\theta} + \cot\phi(\tilde{N}_\phi - \tilde{N}_\theta) + \tilde{Q}_\phi + Q_{\phi c}/2 + \\ &\quad + [\bar{N}_{\phi,\phi} + \operatorname{cosec}\phi\, \bar{N}_{\theta\phi,\theta} + \cot\phi(\bar{N}_\phi - \bar{N}_\theta) + \bar{Q}_\phi]\, s/R = \\ &\quad = \rho hR(s_1\tilde{v} + s_3\bar{v} + s_4 R\beta_\phi)_{,tt},\end{aligned}$$

$$\begin{aligned}&\operatorname{cosec}\phi\, \bar{N}_{\theta,\theta} + \bar{N}_{\theta\phi,\phi} + 2\cot\phi\, \bar{N}_{\theta\phi} + \bar{Q}_\theta - Q_{\theta c}R/2c + \\ &\qquad + (\operatorname{cosec}\phi\, \tilde{N}_{\theta,\theta} + \tilde{N}_{\theta\phi,\phi} + 2\cot\phi\, \tilde{N}_{\theta\phi} + \tilde{Q}_\theta)s/R = \\ &\qquad = \rho hR(s_3\tilde{u} + s_2\bar{u} + s_5 R\beta_\theta)_{,tt},\end{aligned}$$

$$\begin{aligned}&\bar{N}_{\theta,\phi} + \operatorname{cosec}\phi\, \bar{N}_{\theta\phi,\theta} + \cot\phi\,(\bar{N}_\phi - \bar{N}_\theta) + \bar{Q}_\phi - Q_{\phi c}R/2c + \\ &\quad + [\tilde{N}_{\phi,\phi} + \operatorname{cosec}\phi\, \tilde{N}_{\theta\phi,\phi} + \cot\phi\,(\tilde{N}_\phi - \tilde{N}_\theta) + \tilde{Q}_\phi] = \\ &\quad = \rho hR(s_3\tilde{v} + s_2\bar{v} + s_5 R\beta_\phi)_{,tt},\end{aligned}$$

$$\begin{aligned}&(\tilde{Q}_\phi + Q_{\phi c}/2)_{,\phi} + \operatorname{cosec}\phi(\tilde{Q}_\theta + Q_{\theta c}/2)_{,\theta} + \cot\phi\,(\tilde{Q}_\phi + Q_{\phi c}/2) - \\ &\quad - (\tilde{N}_\phi + \tilde{N}_\theta) + [\bar{Q}_{\phi,\phi} + \operatorname{cosec}\phi\, \bar{Q}_{\theta,\theta} + \cot\phi\, \bar{Q}_\phi - (\bar{N}_\phi + \bar{N}_\theta)]\, s/R = \\ &\quad = \rho hRs_1 w_{,tt},\end{aligned}$$

$$\begin{aligned}&\operatorname{cosec}\phi\, \tilde{M}_{\theta,\theta} + \tilde{M}_{\theta\phi,\phi} + 2\cot\phi\, \tilde{M}_{\theta\phi} - R(\tilde{Q}_\theta + \bar{Q}_\theta s/R) + Q_{\theta c}hR/4c + \\ &\qquad + [\operatorname{cosec}\phi\, \bar{M}_{\theta,\theta} + \bar{M}_{\theta\phi,\phi} + 2\cot\phi\, \bar{M}_{\theta\phi} - R(\bar{Q}_\theta + \tilde{Q}_\theta s/R)]\, s/R = \\ &\qquad = \rho hR^2(s_4\tilde{u} + s_5\bar{u} + s_6 R\beta_\theta)_{,tt},\end{aligned}$$

$$\begin{aligned}&\tilde{M}_{\phi,\phi} + \operatorname{cosec}\phi\, \tilde{M}_{\theta\phi,\theta} + \cot\phi\,(\tilde{M}_\phi - \tilde{M}_\theta) - R(\tilde{Q}_\phi + \bar{Q}_\phi s/R) + Q_{\phi c}hR/4c + \\ &\quad + [\bar{M}_{\phi,\phi} + \operatorname{cosec}\phi\, \bar{M}_{\theta\phi,\theta} + \cot\phi(\bar{M}_\phi - \bar{M}_\theta) - R(\bar{Q}_\phi + \tilde{Q}_\phi s/R)]s/R = \\ &\quad = \rho hR^2(s_4\tilde{v} + s_5\bar{v} + s_6 R\beta_\phi)_{,tt}.\end{aligned} \tag{3.7}$$

After application of the variation technique, the following stress-strain equations in addition to eqs. (3.7) are obtained

$$\begin{aligned}&\tilde{N}_\phi + \bar{N}_\phi s/R = K(\tilde{\varepsilon}_\phi + \nu\tilde{\varepsilon}_\theta)/R, \quad \bar{N}_\phi + \tilde{N}_\phi s/R = K(\bar{\varepsilon}_\phi + \nu\bar{\varepsilon}_\theta)/R, \\ &\tilde{N}_{\theta\phi} + \bar{N}_{\theta\phi} s/R = Gh\tilde{\gamma}_{\theta\phi}/R, \quad \bar{N}_{\theta\phi} + \tilde{N}_{\theta\phi} s/R = Gh\bar{\gamma}_{\theta\phi}/R, \\ &\tilde{M}_\phi + \bar{M}_\phi s/R = D(k_\phi + \nu k_\theta)/R, \quad \tilde{M}_{\theta\phi} + \bar{M}_{\theta\phi} s/R = D(1-\nu)k_{\theta\phi}/2R, \\ &\tilde{Q}_\phi + \bar{Q}_\phi s/R = \tfrac{5}{6}Gh(r_1\tilde{\gamma}_{r\phi} + r_3\gamma_{r\phi c})/R, \quad \bar{Q}_\phi + \tilde{Q}_\phi s/R = \tfrac{5}{6}Ghr_2\bar{\gamma}_{r\phi}/R, \\ &Q_{\phi c} = \tfrac{5}{3}Gh(r_3\tilde{\gamma}_{r\phi} + r_4\gamma_{r\phi c})/R.\end{aligned} \tag{3.8}$$

The other six relationships for $\tilde{N}_\theta$, $\overline{N}_\theta$, $\tilde{M}_\theta$, $\tilde{Q}_\theta$, $\overline{Q}_\theta$ and $Q_{\theta c}$, not appearing in eqs. (3.8), can be written by interchanging ϕ and θ. While obtaining the shear stress resultants, $\tilde{Q}_\phi$, $\overline{Q}_\phi$, $Q_{\phi c}$ etc., the shear stresses at the interfaces of the core and the face sheets are eliminated. The quantities r_1, r_2, ..., etc. are given by the following equations:

$$r_1 = 1 + r_G/(15r_G+20r_h)\,, \quad r_2 = 1 + r_G/(15r_G+40r_h)\,,$$
$$r_3 = 2r_G r_h/(15r_G+20r_h)\,, \quad r_4 = 2r_G r_h(15r_G+24r_h)/(15r_G+20r_h)\,. \tag{ii}$$

4. Solution of the differential equations

For normal modes of vibration the displacement components $\tilde{u}$, $\bar{u}$, $\tilde{v}$, $\bar{v}$, w, β_ϕ and β_θ are expressed as

$$(\tilde{u}, \bar{u}, \tilde{v}, \bar{v}, w, R\beta_\phi, R\beta_\theta) = (\tilde{U}, \overline{U}, \tilde{V}, \overline{V}, W, X, Y)\exp(j\omega t)\,, \tag{4.1}$$

where ω is the circular frequency of the sandwich shell and $j=(-1)^{\frac{1}{2}}$.

At this stage, a set of new functions $\tilde{H}$, $\overline{H}$, $\tilde{\chi}$, $\tilde{\Gamma}$, $\bar{\Gamma}$ are introduced, which are defined by

$$\tilde{H} = \tilde{V}_{,\phi} + \cot\phi\,\tilde{V} + \operatorname{cosec}\phi\,\tilde{U}_{,\theta}\,, \quad \overline{H} = \overline{V}_{,\phi} + \cot\phi\,\overline{V} + \operatorname{cosec}\phi\,\overline{U}_{,\theta}\,,$$
$$\tilde{\chi} = \tilde{U}_{,\phi} + \cot\phi\,\tilde{U} - \operatorname{cosec}\phi\,\tilde{V}_{,\theta}\,, \quad \bar{\chi} = \overline{U}_{,\phi} + \cot\phi\,\overline{U} - \operatorname{cosec}\phi\,\overline{V}_{,\theta}\,,$$
$$\tilde{\Gamma} = X_{,\phi} + \cot\phi\,X + \operatorname{cosec}\phi\,Y_{,\theta}\,, \quad \bar{\Gamma} = Y_{,\phi} + \cot\phi\,Y - \operatorname{cosec}\phi\,X_{,\theta}\,. \tag{iii}$$

As will be seen later, the use of these special functions considerably simplifies the differential equations of motion. Replacing eqs. (3.8), (4.1), (iii) together with the nondimensional frequency parameter $\Omega = (\rho\omega^2 R^2/E)^{\frac{1}{2}}$ in eqs. (3.7), results in the following system of equations

$$A_1\tilde{H}_{,\phi} - \operatorname{cosec}\phi\,\tilde{\chi}_{,\theta} + A_2\tilde{V} + A_3\overline{V} + A_4 X + A_5 W_{,\phi} = 0\,,$$
$$A_1\operatorname{cosec}\phi\,\tilde{H}_{,\theta} + \tilde{\chi}_{,\phi} + A_2\tilde{U} + A_3\overline{U} + A_4 Y + A_5\operatorname{cosec}\phi\,W_{,\theta} = 0\,,$$
$$A_1\overline{H}_{,\phi} - \operatorname{cosec}\phi\,\tilde{\chi}_{,\theta} + A_3\tilde{V} + A_6\overline{V} + A_7 X + A_8 W_{,\phi} = 0\,,$$
$$A_1\operatorname{cosec}\phi\,\tilde{H}_{,\theta} + \tilde{\chi}_{,\phi} + A_3\tilde{U} + A_6\overline{U} + A_7 Y + A_8\operatorname{cosec}\phi\,W_{,\theta} = 0\,,$$
$$B_1\tilde{H} + B_2\overline{H} + B_3\tilde{\Gamma} + B_4\nabla^2 W + B_5 W = 0\,,$$
$$A_1\tilde{\Gamma}_{,\phi} - \operatorname{cosec}\phi\,\bar{\Gamma}_{,\theta} + B_6\tilde{V} + B_1\overline{V} + B_8 X + B_9 W_{,\phi} = 0\,,$$
$$A_1\operatorname{cosec}\phi\,\tilde{\Gamma}_{,\theta} + \bar{\Gamma}_{,\phi} + B_6\tilde{U} + B_7\overline{U} + B_8 Y + B_9\operatorname{cosec}\phi\,W_{,\theta} = 0\,. \tag{4.2}$$

In the above equations ∇^2 is Laplace operator in spherical coordinates

$$\nabla^2 = \partial^2/\partial\phi^2 + \cot\phi\,\partial/\partial\phi + \operatorname{cosec}\phi\,\partial^2/\partial\theta^2\,, \tag{iv}$$

and the quantities A_1, A_2, ..., B_9 are

$$A_1 = 2/(1-\nu)\,, \quad A_2 = 2 + 2\Omega^2 s_1(1+\nu) - 5B_4/6\,,$$
$$A_3 = 2\Omega^2 s_3(1+\nu) + 5B_2/6\,, \quad A_4 = 2\Omega^2 s_4(1+\nu) + 5B_3/6\,,$$
$$A_5 = A_1(1+\nu) + 5B_4/6\,, \quad A_6 = 2 + 2\Omega^2 s_2(1+\nu) - \tfrac{5}{6}(r_2 + r_4 R^2/c^2)\,,$$
$$A_7 = 2\Omega^2 s_5(1+\nu) + \tfrac{5}{6}\{r_2 s/R - (r_3 - r_4/2r_h)R/c\}\,,$$
$$A_8 = -(r_3+r_4)5R/6c\,, \quad B_1 = -(17+7\nu)/5(1-\nu)\,,$$
$$B_2 = (r_3+r_4)R/c\,, \quad B_3 = r_1 + r_3 - (r_3+r_4)/2r_h\,,$$
$$B_4 = r_1 + 2r_3 + r_4\,, \quad B_5 = \{\Omega^2 s_1 - 2/(1-\nu)\}\tfrac{12}{5}(1+\nu)\,,$$
$$B_6 = [10B_3 + 24(1+\nu)\Omega^2 s_4]R^2/h^2\,,$$
$$B_7 = [24(1+\nu)\Omega^2 s_5 - 10\{(r_3 - r_4/2r_h)R/c - r_2 s/R\}]R^2/h^2\,,$$
$$B_8 = [24(1+\nu)\Omega^2 s_6 - 10\{r_1 - r_3/r_h + r_4/4r_h^2 + r_2 s^2/R^2\}]R^2/h^2\,,$$
$$B_9 = -10B_3 R^2/h^2\,. \tag{v}$$

Equations (4.2) can be further simplified to a more suitable form, from where the solution can be directly obtained:

$$\begin{aligned}
&(A_1\nabla^2+A_2)\tilde{H}+A_3\overline{H}+A_4\tilde{\Gamma}+A_5\nabla^2 W=0\,,\\
&A_3\tilde{H}+(A_1\nabla^2+A_6)\overline{H}+A_7\tilde{\Gamma}+A_8\nabla^2 W=0\,,\\
&B_1\tilde{H}+B_2\overline{H}+B_3\tilde{\Gamma}+(B_4\nabla^2+B_5)W=0\,,\\
&B_6\tilde{H}+B_7\overline{H}+(A_1\nabla^2+B_8)\tilde{\Gamma}+B_9\nabla^2 W=0\,,
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
&(\nabla^2+A_2)\tilde{\chi}+A_3\bar{\chi}+A_4\bar{\Gamma}=0\,,\\
&A_3\tilde{\chi}+(\nabla^2+A_6)\bar{\chi}+A_7\bar{\Gamma}=0\,,\\
&B_6\tilde{\chi}+B_7\bar{\chi}+(\nabla^2+B_8)\bar{\Gamma}=0\,.
\end{aligned}\tag{4.4}$$

The general solution of eqs. (4.3) and (4.4) is expressed in terms of associated Legendre functions of the first and second kinds.

$$\begin{aligned}
&\tilde{H}=\sum_{m=0}^{\infty}\sum_{\alpha=1}^{4}\tilde{H}_\alpha^m\cos m\theta\,,\quad \overline{H}=\sum_{m=0}^{\infty}\sum_{\alpha=1}^{4}\overline{H}_\alpha^m\cos m\theta\,,\\
&\tilde{\Gamma}=\sum_{m=0}^{\infty}\sum_{\alpha=1}^{4}\tilde{\Gamma}_\alpha^m\cos m\theta\,,\quad W=\sum_{m=0}^{\infty}\sum_{\alpha=1}^{4}W_\alpha^m\cos m\theta\,,
\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}
&\tilde{\chi}=\sum_{m=0}^{\infty}\sum_{\beta=1}^{3}\tilde{\chi}_\beta^m\cos m\theta\,,\quad \bar{\chi}=\sum_{m=0}^{\infty}\sum_{\beta=1}^{3}\bar{\chi}_\beta^m\cos m\theta\,,\\
&\bar{\Gamma}=\sum_{m=0}^{\infty}\sum_{\beta=1}^{3}\bar{\Gamma}_\beta^m\cos m\theta\,,
\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}
&\tilde{H}_\alpha^m=A_\alpha^m p_{\nu_\alpha}^m(\cos\phi)+B_\alpha^m Q_{\nu_\alpha}^m(\cos\phi)\,,\quad \overline{H}_\alpha^m=C_\alpha^m p_{\nu_\alpha}^m(\cos\phi)+D_\alpha^m Q_{\nu_\alpha}^m(\cos\phi)\,,\\
&\tilde{\chi}_\beta^m=M_\beta^m P_{\eta_\beta}^m(\cos\phi)+N_\beta^m Q_{\eta_\beta}^m(\cos\phi)\,,\quad \text{etc.}
\end{aligned}\tag{vi}$$

5. Boundary conditions

In order to have the complete formulation of the problem, boundary conditions are to be introduced. A generalized system of boundary conditions is obtained from the variation of the energy equation for the following quantities:

$$\begin{aligned}
&\tilde{N}_\phi+\overline{N}_\phi s/R \text{ or } \tilde{V}\,, \qquad \overline{N}_\phi+\tilde{N}_\phi s/R \text{ or } \overline{V}\,,\\
&\tilde{N}_{\theta\phi}+\overline{N}_{\theta\phi} s/R \text{ or } \tilde{U}\,, \qquad \overline{N}_{\theta\phi}+\tilde{N}_{\theta\phi} s/R \text{ or } \overline{U}\,,\\
&\tilde{M}_\phi+\overline{M}_\phi s/R \text{ or } \beta_\phi\,, \qquad \tilde{M}_{\theta\phi}+\overline{M}_{\theta\phi} s/R \text{ or } \beta_\theta\,,\\
&\tilde{Q}_\phi+\overline{Q}_\phi s/R+Q_{\phi c}/2 \text{ or } w\,.
\end{aligned}\tag{5.1}$$

However, in the problem of vibration of a shell fixed at the boundary $\phi=\phi_0$, relevant boundary conditions which are only in terms of displacements are prescribed.

6. Numerical results and discussion

This section describes the results for the case of axisymmetric vibration of a deep sandwich spherical shell. For this case the variables involved in the analysis become independent of the parameter θ and as such the equations are simplified. Further, the displacement components $\tilde{U}$, $\overline{U}$ and Y which are defined by eq. (4.1) are zero. It is also seen that the system of equations (4.4) is identically satisfied.

The values of the associated Legendre functions $P_{\nu_\alpha}^m(\cos\phi)$ for real and complex orders ν_α are not available in the literature. The authors have generated the values of the Legendre tions $P_{\nu_\alpha}(\cos\phi)$ and their derivatives with respect to ϕ for real and complex orders ν_α. se tables have been compiled and reported earlier [8]. It can be seen that for the axisym-ric vibration the form of the Legendre functions used in the general solution given in eqs. is further simplified as $m=0$. The coefficients of the Legendre functions $Q_{\nu_\alpha}(\cos\phi)$ of the nd kind are zero due to the finiteness condition on the solution at $\phi=0$.

he homogenous boundary conditions at the clamped edge $\phi=\phi_0$ are written as

$$\tilde{V}=0\,,\quad \bar{V}=0\,,\quad W=0\,,\quad X=0\,. \tag{6.1}$$

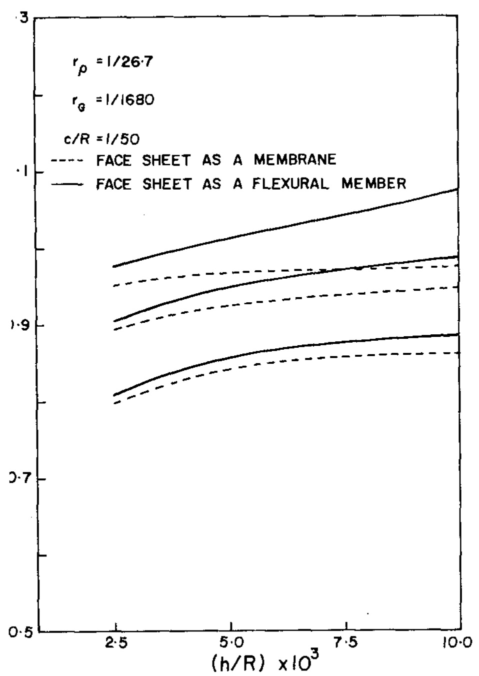
frequency equation obtained by using eq. (6.1) is a transcendental equation. The roots of equation are generated by an iteration process and all computations were performed on an 1 360. This technique has been used earlier [6] to generate other results. The numerical ies of the frequency parameter Ω for the clamped sandwich spherical shell have been com-d for the following non-dimensionalized material properties:

$$r_\rho = 1/26.7\,,\quad c/R = 0.02\,,\quad r_G = 1/1680\,,$$

. Poisson's ratio of the face sheet $\nu=0.3$. The above properties are valid assumptions for lls with aluminum face sheets and aluminum honeycomb as the core material.

'he frequency ω in its non-dimensionalized form, which is obtained by dividing ω by the :or $R^{-1}(E/\rho)^{\frac{1}{2}}$, has been plotted as a function of h/R in Figures 2–4. These graphs are shown various ϕ values varying from 60° to 120° at intervals of 30°. The frequency Ω, as expected, reases with the increase in h/R for all values of angles ϕ_0 as well as the modes. The nature hese curves agrees with the trend of variation of Ω for homogeneous shells reported earlier he literature [1]. It is seen that as the angle ϕ_0 increases the frequency Ω gradually goes down. is is partly explained by the predominance of the bending effects in the face sheets for lower ues of opening angles ϕ_0. Another effect to be noticed from the curves is the sharp sensitivity Ω to the ratio h/R, which is a property of the face sheets, for lower values of ϕ_0.

The frequency parameter Ω has also been calculated for the pinned edge conditions at $=\phi_0$. These conditions are



;ure 2. Frequency variation with h/R for clamped edge condition at $\phi_0=60°$ for three modes.

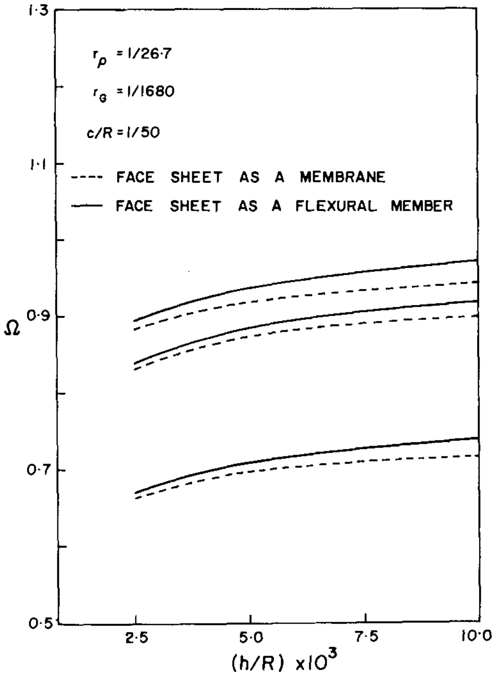


Figure 3. Frequency variation with h/R for clamped edge condition at $\phi_0 = 90°$ for three modes.

$$\tilde{V} = 0, \quad \bar{V} = 0, \quad W = 0, \quad \tilde{M}_\phi + \bar{M}_\phi s/R = 0. \tag{6.2}$$

The detailed numerical results for this particular case are not presented in this paper.

The frequency values for these two cases are closer to each other for thin face sheets. As the face sheet grows thicker i.e., for larger values of h/R, the discrepancy between the Ω values as generated by the use of conditions (6.1) and (6.2) is more pronounced. This behavior is well

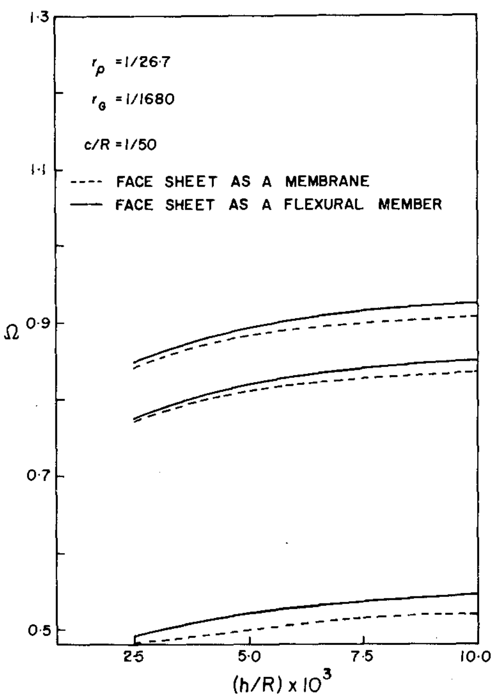


Figure 4. Frequency variation with h/R for clamped edge condition at $\phi_0 = 120°$ for three modes.

explained from the fact that for thinner face sheets, which correspond to lower h/R values, the membrane action is more predominant as compared to the bending action.

In order to study the effects of bending in the face sheets, extensive computations were performed for the two cases of bending and membrane face sheets. The results for sandwich shells with membrane face sheets for $\phi_0 = 60°$, $90°$ and $120°$, shown by broken lines in Figs. 2, 3 and 4, are obtained by making a proper simplification of the general frequency equation presented in this paper. By introducing necessary changes in the material and geometric properties of the sandwich shell discussed in reference [6] and carrying out the computations, it was found that the results obtained here for the membrane face sheet case are in close agreement with those in [6]. It is observed that the error involved in the frequence Ω by the neglect of the bending behavior of the face sheet increased as ϕ_0 decreases. Further, this error increases for higher modes. The maximum error occurs for the third mode in the case of $\phi_0 = 60°$ and its magnitude is 9.3 %. This error goes down to 1.6 % for $\phi_0 = 120°$. From the study of the graphs presented here, it can be concluded that for shells with lower ϕ_0, the effect of curvature decreases for higher modes. This is in agreement with one of the observations made in [4] for shallow shells. No computations are performed for the case with no rotary inertia in the core. It is, however, expected that rotary inertia will play a significant role for higher modes.

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